GENERALIZING DOMAIN AND RANGE FROM SINGLE-VARIABLE TO MULTIVARIABLE FUNCTIONS

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The purpose of this paper is to describe (a) multivariable calculus students' meanings for the domain and range of single and multivariable functions and (b) how they generalize their meanings for domain and range from single-variable to multivariable functions. We first describe how students think about domain and range of multivariable functions as inputs and outputs, independent and dependent quantities, and as associated with particular variables. We then use an actor-oriented transfer framework to describe the ways in which students identify similarities between domain and range in single- and multivariable functions, and how they use these similarities to generalize their meanings for domain and range.

INTRODUCTION

While researchers have identified interesting and useful phenomena about how students think about single-variable functions, far fewer studies exist about how these findings might extend to multivariable functions This motivates the first focus of our paper. Multivariable functions form the backbone of multivariable calculus, and are frequently used in physics and other sciences, but research about how students understand multivariable functions and ideas in multivariable calculus is largely preliminary (Kabael, 2011; Martinez-Planell & Trigueros, 2013; Trigueros & Martinez-Planell, 2010; Yerushalmy, 1997). Given the documented difficulties students have with single-variable functions and single-variable calculus, it bears investigating if and how these difficulties appear in multivariable functions and multivariable calculus. We focus on domain and range because researchers have suggested that a robust conception of function begins with students thinking about the correspondence between inputs and outputs; that is, the function's domain and range (Oehrtman, Carlson, & Thompson, 2008).

It is clear to experts that multivariable functions and ideas related to them (e.g., domain, range, rate of change) are extensions of the same ideas in the single-variable function context. However, students do not always make the connections that experts do, and they do not necessarily develop the meanings that instructors intend. This motivates the second focus of our paper. We analyze what students see as similar between the domain and range of single- and multivariable functions. This actor-oriented perspective yields insight into how students generalize ideas; that is, how they develop meanings for ideas in a novel setting by leveraging their meanings from a familiar setting. Though there have been many studies about generalization in algebra (e.g. Amit & Klass-Tsirulnikov, 2005; Carpenter & Franke, 2001; Cooper &

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Warren, 2008; Ellis, 2007), there are fewer in undergraduate mathematics topics. At the same time, generalization is a critical component of mathematical thinking (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Mason, 1996; Peirce, 1902; Sriraman, 2003; Vygotsky, 1986) and thus it is important to extend knowledge of how students generalize in higher mathematics. If we know the specific ways in which students generalize their ideas about single-variable functions to multivariable functions, instructors can build on connections that appear naturally to students while providing evidence to counter any unproductive generalizations (that is, not congruent with experts' views) students make.

BACKGROUND LITERATURE

While a systematic search of the literature did not reveal studies explicitly focused on domain and range, there are some findings in the function literature that are relevant to the present study. For instance, one way to define domain and range is the set of inputs and outputs of the function, respectively. According to Oehrtman, Carlson, and Thompson (2008), thinking about a function in terms of an input and corresponding output is the beginning of a robust function conception. Monk (1994) found that most calculus students have developed this pointwise view of function but fewer develop an across-time view of function, in which students' conception of function progress to thinking about the function for infinitely many values and understanding how the a change in one variable affects the other(s). That is, a robust function conception involves not only the ability to pair an input with an output, but an understanding of the relationship between quantities. Confrey and Smith (1995) say the beginning of this understanding occurs as students form connections between values in a function's domain and range. However, as function is introduced in algebra and/or precalculus, the functions instructors ask students to reason about are single-variable functions. How students build an understanding of multivariable functions is not known. Investigating students' meanings for domain and range thus extends the literature about students' understanding of single-variable functions, and adds to the body of knowledge that has just begun to develop regarding students' understanding of functions of more than one variable.

When students learn multivariable functions, they must broaden their notion of function beyond the single-variable case; that is, they must generalize their ideas. Note that abstraction is also a key part of this process, but space limits the discussion to generalization. Generalization is a critical component of mathematical thinking (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Mason, 1996; Peirce, 1902; Sriraman, 2003; Vygotsky, 1986), and while there have been many studies about generalization in algebra (e.g. Amit & Klass-Tsirulnikov, 2005; Carpenter & Franke, 2001; Cooper & Warren, 2008; Ellis, 2007), but far fewer studies exist about generalization in undergraduate mathematics. Findings from generalization studies typically indicate that generalization is difficult for students; for instance, algebra students' over-generalize of linear relationships interferes with their understanding of quadratic, exponential, and logarithmic functions (Chazan, 2006; Ellis & Grinstead, 2008;

Schwarz & Hershkowitz, 1999; Zaslavsky, 1997). Other difficulties include trouble transitioning from pattern generalization to abstract algebraic thinking (e.g., Moss, Beatty, McNab, & Eisenband, 2006; Mason, 1996; Orton & Orton, 1999; Schliemann, Carraher, & Brizuela, 2001) and shifting from thinking about a pattern recursively to developing a formula for the n^{th} case. If we know the specific ways in which students generalize, instructors can build on connections that appear naturally to students while providing evidence to counter any unproductive generalizations (that is, not congruent with experts' views) students make.

THEORETICAL PERSPECTIVE

We studied generalization from an actor-oriented perspective, which attends to what students see as similar in mathematical situations. This is in contrast to an observer-oriented perspective in which students' ideas are compared to what an expert would see as similar across situations. Such perspectives often find that students cannot or do not generalize ideas from one setting to another, and focus on students' final generalizations rather than generalization as a process. We are interested in *how* students generalize, and the actor-oriented perspective allows us to privilege students' perceptions of similarity, even if those perceptions are not necessarily correct. We follow Ellis (2007) and Lobato (2003) in thinking about generalization as "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). This was a useful lens for looking at how students viewed domain and range, a topic they had experienced prior with single-variable functions, in the novel situation of multivariable functions. We use Ellis' (2007) generalizations taxonomy as an analytic framework, which is detailed later in the paper.

DATA COLLECTION AND ANALYSIS

We interviewed 20 students enrolled in multivariable calculus at a mid-size university in the northwestern U.S. The students were volunteers from all the multivariable calculus students enrolled during that term, and were compensated for their participation. The course topics included vectors, vector functions, curves in two and three dimensions, surfaces, partial derivatives, gradients, directional derivatives, and multiple integrals in different coordinate systems. Each student participated in a semi-structured interview that lasted about an hour. We recorded audio and written work from each of the interviews using a LiveScribe Echo Pen, which provides a recording consisting of synced audio and written work of the student. These recordings also allowed us to create dynamic playbacks of the interviews during analysis of the data.

The students responded to the following tasks, which were developed to elicit their verbal definitions for the concepts (Q1) and how they operationalized those definition in problem contexts involving single-variable (Q2) and multivariable (Q3, Q4) functions.

Q1. What does domain mean? What does range mean?

Q2. What are the domain and range of $f(x)=4+1/(x-3)$?

Q3. What are the domain and range of $f(x,y)=x^2+y^2$?

Q4. What are the domain and range of $x^2+y^2+z^2=9$?

Each research foci required its own analytic framework. We used a constant comparative analysis Corbin (2008) to investigate students' meanings for domain and range (the first focus), and Ellis' (2007) generalizations taxonomy to investigate how students generalized their meanings for domain and range (the second focus). We discuss the specifics of the constant comparative analysis and its results in the next section, and the specifics of the generalizations analysis and its results in the section following that.

MEANINGS FOR DOMAIN AND RANGE

To perform the constant comparative analysis, Researcher 1 listened to half of the interviews and highlighted phrases students used to talk about domain and range. These phrases included words like input, dependent variable, 'goes with x,' etc. The researcher formed codes from these words (e.g., input/output, independence/dependence, associated with particular variables) and used these to code the second half of the data. The researcher added to and modified the codes based on this data, and then both researchers independently used the codes to code all of the data. They compared results, discussed any differences, and modified the codes a final time. The researchers then used the coded and categorized data to describe students' meanings for domain and range.

These meanings fit into three categories: as attached to specific variables (e.g., Adam), input/output (e.g., Jim), and independence/dependence (e.g., Phillip). We found that students talked about all of these ideas for both single-variable and multivariable functions, as is evident in the selected excerpts below.

Adam: [Q3] It's a helix, or spinny spring looking thing. Domain and range, so the domain of this would be all real numbers for x values, so x can equal any number, and it changes what z equals, but even negative numbers squared equal positive z. And the range is all real numbers because there is no value of y for which the graph is undefined.

Jim: [Q1] Domain is your input values, otherwise known as your x values. It could also represent your independent values. The range is your output, your dependent variables, y values.

[Q3] There would be two different domains. You have your x input and our y input. Your x domain and your y domain give you a range of a different variable. It's the range of z or $f(x,y)$.

Phillip: [Q3] It's a function of two variables. x and y are both independent variables, rather than the dependent variable. You could say the domain is the independent variable and range is the dependent variable.

Students who thought about domain and range as attached to specific variables thought that domain always meant the possible values of x and range meant the possible value of y, regardless of whether the function was $f(x)$ or $f(x,y)$. Other students' meanings for domain and range relied on the notion of function as a 'machine' which generates inputs from outputs; these students' meaning for domain was the possible input values and their meaning for range was the possible output values. Finally, students thought of domain as a set of values for an independent variable, and range as a set of values for a dependent variable. These categories are not mutually excusive, and many students had a meaning for domain and range that incorporated both ideas of input/output and independence/dependence. Having identified students' particular meanings, we then analysed the generalization of those meanings from the single-variable to multivariable context.

GENERALIZING THE MEANING OF DOMAIN AND RANGE

The generalization analysis was based on Ellis' (2007) generalizations taxonomy (see Ellis, 2007, p. 235, 245). The taxonomy distinguishes between *generalizing actions*, which are "learners' mental acts as inferred through the person's activity and talk" (Ellis, 2007, p. 233) and *reflection generalizations*, or students' public statements about a property or pattern common to two situations. The taxonomy includes subcategories that represent specific types of generalizing actions and reflection generalizations. The first researcher coded all of the data according to the descriptions indicated in the tables. The second researcher reviewed the coding and any the researchers discussed and adjudicated any points of disagreement. Not all of the categories in Ellis' (2007) taxonomy appeared in this data. The categories that did fit, their descriptions, and examples from this data are shown in tables 2 and 3.

We found that students primarily used the generalization methods of relating objects (equations and graphs), stating global rules, and using and/or modifying prior ideas and strategies. Students often appealed to the similarities of $f(x)$ and $f(x,y)$ as each being a function "of" something, and stating that in the multivariable case, x,y were inputs or independent variables just as x was an input in the single-variable case. They used this to justify that the domain of $f(x,y)$ was the possible values for x and y, as it was in the single-variable case. Students related graphs by noticing that the range typically had to do with the variable on the vertical axis and domain typically had to do on with the variable on the horizontal axis, and they used this to infer that the range of a multivariable function would be the possible z values and the domain would be for the horizontal plane. Finally, other students stated that domain would mean the input or independent variable no matter how many variables in the function argument, and range would always be the output, the dependent value, or the function value. These

similarities allowed them to apply a prior idea or modify their idea as they thought about the domain and range of multivariable functions.

Ellis (2007) framework			Example in domain/range data
Type I: Relating	1. Relating situations: The formation of an association between two or more problems or situations.	Connecting Back: The formation of a connection between a current situation and a previously-encountered situation.	Domain is your input values. It could also represent your independent values. I am trying to think like in terms of my physics lab where there are independent and dependent variables and you plug in the numbers that you use.
		Creating New: The invention of a new situation viewed as similar to an existing situation.	Say you need to calculate temperature and you have the temperature relative to California and you have some conversion, so the input values are the temperatures in Oregon and the output values are the temperature in California.
	2. Relating objects: The formation of an association between two or more present objects.	Property: The association of objects by focusing on a property similar to both.	Lets call z the dependent variable here and move the x and y to the other side. Now the domain is x and y.
		Form: The association of objects by focusing on their similar form.	You can't have negative z but I don't know if that's the domain or the range. I'm going to say it's the range, and treat the z axis like the y axis of the function.
Type III: Extending	1. Expanding the range of Applicability: The application of a phenomenon to a larger range of cases than that from which it originated.		Domain is your input values, otherwise known as your x values. It could also represent your independent values. The range is your output, your dependent values, your y values.
	2. Removing Particulars: The removal of some contextual details in order to develop a global case.		I am a little fuzzy on range in 3D. I think in 2 dimensions, whatever your domain is, you put that in and that's what your output is. I suppose that's the same in 3D as well: the array of possible values I can get out of the function.

Table 1: Generalizing actions for domain and range.

Ellis (2007) framework			Example in domain/range data
Type IV: Identification or Statement	3. General Principle: A statement of a general phenomenon.	Rule: The description of a general formula or fact.	[Q3] Domain of this would be all real numbers for *x* values, so *x* can equal any number, and it changes what *z* equals, but even negative numbers squared equal positive *z*. And the range is all real numbers because there is no value of *y* for which the graph is undefined.
		Global Rule: The statement of the meaning of an object or idea.	[Q3] *Z* is kind of like the function value. It equals *f*(x,y) kind of like *y* = *f*(x). It's the dependent variable, not the independent.
Type VI: Influence	1. Prior Idea or Strategy: The implementation of a previously-developed generalization.		[Q1] Range is the set of numbers the function can have. [Q4] I think the range is 9 for this one… because that's the value on the other side of the equal sign. So it can't range to any other values.
	2. Modified Idea or Strategy: The adaptation of an existing generalization to apply to a new problem or situation.		[Q3] In this instance the range is *z*, the output value. So I would say the variables applied to the function doesn't necessarily correspond to domain as *x*, range as *y*. So if I looked back to my definitions in question one, I could define domain and range in 3D space with domain as the span of values that can occur on the horizontal plane and I would define range to be the span of values that are dependent on the domain and span the vertical plane.

Table 2: Reflection generalizations for domain and range.

IMPLICATIONS FOR INSTRUCTION AND SUGGESTIONS FOR FURTHER RESEARCH

The results of actor-oriented generalization research have direct implications for instruction. Knowing what students see as similar allows instructors to build on the productive connections that appear naturally to students. For instance, many of our subjects developed a mathematically correct notion of the domain of $f(x,y)$ by thinking of $f(x,y)$ as having two inputs (and a domain for each) just as $f(x)$ has one input (and a corresponding domain). Others thought about extending the concept of an independent variable (and its domain) to two independent variables (with a domain for each). Therefore, instructors can introduce multivariable functions by referencing students' notions of inputs, outputs, independence, and dependence. They can also point out the generalizations students may make that are not mathematically correct, such as explicitly noting that 'domain is *x*, range is *y*' is not necessarily correct for functions of more than one variable.

Our further research plans are to select other topics in multivariable calculus, such as partial derivatives and multiple integrals, and study both students' understanding of these concepts and their generalizations from single- to multivariable calculus.

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